LINES ON NON-DEGENERATE SURFACES

GUANGFENG JIANG AND MUTSUO OKA

Abstract. On an affine variety X defined by homogeneous polynomials, every line in the tangent cone of X is a subvariety of X. However there are many other germs of analytic varieties which are not of cone type but contain "lines" passing through the origin. In this paper, we give a method to determine the existence and the "number" of such lines on non-degenerate surface singualrities.

1. Introduction

Let (X, O) be a germ of analytic varieties embedded in $(\mathbb{C}^n, O)$ with a singularity at O. By abuse of language, we say that L is *a line* in (X, O) if (L, O) is a smooth curve germ in (X, O) and $L \setminus \{0\}$ is contained in the regular part of X.

In [3, 5], lines on hypersurfaces with simple singularities are classified by using the classification machinery. All the hypersurfaces of dimension 2 and 3 with simple or simple elliptic singularities passing through x-axis are equivalent to (under the coordinate transformation preserving the x-axis) some surfaces defined by explicit equations. It turns out that the A, D, E singularities split in this classification. This says that different smooth curves on the same surface might have different properties.

Let $\pi : \tilde{X} \to (X, O)$ be a resolution of a surface (X, O) with an isolated singularity at the origin O and let $\{E_1, \dots, E_r\}$ be the exceptional divisors of π. For an exceptional divisor E_i, let $\mathcal{L}_{E_i}$ denote the set of lines on $(X, 0)$ whose strict transform intersect E_i transversally. It is known that $\mathcal{L}_{E_i}$ is non-empty if and only if there exist a function germ h in the maximal ideal $\mathfrak{m}$ such that the multiplicity of $\pi^* h$ along E_i is one and conversely any line in X is contained in some $\mathcal{L}_{E_i}$ ([1, 2]). We call E_i a *normally smooth divisor* if $\mathcal{L}_{E_i} \neq \emptyset$. Geometrically this implies that $d\pi(v) \neq 0$ for any tangent vector $v \in T_P \tilde{X}$ as long as $P \in E_i \setminus \bigcup_{j \neq i} E_j$ and v is not tangent to E_i. If E_i is normally smooth, any germ of a curve intersecting $E_i \setminus \bigcup_{j \neq i} E_j$ transversely defines a line in X. Any two lines in the same $\mathcal{L}_{E_i}$ can be connected by an analytic family of lines in (X, O).

For a given resolution $\pi : \tilde{X} \to X$, we consider the integer $\rho(\pi) := \sharp\{E_i; \mathcal{L}_{E_i} \neq \emptyset\}$. This number depends on the resolution. Put $\rho(X, O)$ to be the minimal value of $\rho(\pi)$. Obviously $\rho(\pi) = \rho(X, O)$ if $\pi : \tilde{X} \to X$ is a minimal resolution. We call $\rho(\pi)$ *the line index of the resolution* $\pi : \tilde{X} \to X$ and we call $\rho(X, O)$ *the line index* of (X, O).

1991 *Mathematics Subject Classification.* 14J17, 32S25, 32S45.

Key words and phrases. line, normally smooth divisor, weighted homogeneous surface, toric resolution.

The first author was supported by JSPS: P98028.

In this paper, we study $\rho(\pi)$ where π is a toric resolution of a non-degenerate surface singularity. Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a surface defined by $f(z_1, z_2, z_3) = 0$ with isolated singularity at the origin. We assume that f is non-degenerate in the sense of the Newton boundary ([7]). Let Σ^* be a regular simplicial cone subdivision of the dual Newton diagram $\Gamma^*(f)$ and let $\pi : X_{\Sigma^*} \to (X, 0)$ be the associated toric resolution. We denote $\rho(\pi)$ by $\rho(\Sigma^*)$ for simplicity. To each vertex $P = {}^t(p_1, p_2, p_3)$ of Σ^*, there corresponds an exceptional divisor $E(P)$ of π, which may have several components. The multiplicity of $\pi^* z_i$ along $E(P)$ is equal to p_i ([9]). Thus by the result of Gonzalez-Sprinberg and Lejeune-Jalabert ([1]), $E(P)$ is normally smooth if and only if $\min(p_1, p_2, p_3) = 1$. We observe that $\rho(\Sigma^*)$ is independent of the choice of Σ^* under certain conditions (see Proposition 6). This allows us to use the canonical toric resolution to determine $\rho(\Sigma^*)$. Note that a toric resolution is not necessarily minimal. So, in general, $\rho(\Sigma^*)$ may be bigger than $\rho(X, O)$ (see Example 28). However to have the equality $\rho(\Sigma^*) = \rho(X, O)$, it is enough that $\pi : X_{\Sigma^*} \to X$ is line-equivalent to the minimal resolution (see § 2 for the definition). The purpose of this paper is to give a method to compute $\rho(\Sigma^*)$.

2. Line-admissible blowing-ups

Let (X, O) be a germ of a surface with an isolated singularity at O. Suppose that we have a good resolution $\pi_1 : X_1 \to X$ and let $E_1, \dots, E_r$ be the exceptional divisors of π_1. Take a divisor E_{i_0} and a point Q on E_{i_0} and let $\pi_Q : \tilde{X}_1 \to X_1$ be the blowing-up at Q and let E_Q be the exceptional divisor of π_Q. The following statements are obvious.

Proposition 1. *Take a function $h \in \mathfrak{m}$ and let m_i be the multiplicity of $\pi_1^* h$ along E_i. Then the multiplicity m_Q of the pull-back $\pi_Q^*(\pi_1^* h)$ along E_Q is the sum of m_i for all i such that $Q \in E_i$. In particular, $m_Q \geq 1$, and $m_Q = 1$ if and only if $m_{i_0} = 1$ and $Q \in E_{i_0} \setminus \bigcup_{i \neq i_0} E_i$.*

Corollary 2. *Under the situation of Proposition 1, E_Q is a normally smooth divisor of the composition $\pi_1 \circ \pi_Q : \tilde{X}_1 \to X$ if and only if E_{i_0} is a normally smooth divisor of $\pi_1 : X_1 \to X$ and Q is contained in $E_{i_0} \setminus \cup_{j \neq i_o} E_j$.*

We call $\pi_Q : \tilde{X}_1 \to X_1$ a *line-admissible* blowing-up if either the center Q is at the intersection of two exceptional divisor or the supporting divisor is not normally smooth. Suppose that we have another good resolution $\pi_2 : X_2 \to X$. We say that $\pi_2 : X_2 \to X$ is *line-equivalent* to $\pi_1 : X_1 \to X$ if there exist a finite chain of resolutions $\pi_i' : Y_i \to X, i = 1, \dots, s$ such that (1) $Y_1 = X_1$ and $\pi_1' = \pi_1$ and $Y_s = X_2$ and $\pi_s' = \pi_2$ and (2) any consecutive resolutions factor by either $\sigma_i : Y_i \to Y_{i+1}$ or $\sigma_i' : Y_{i+1} \to Y_i$, where σ_i and σ_i' are line-admissible blowing-ups.

An immediate consequence of the definition and Corollary 2 is:

Corollary 3. *Assume that $\pi_i : X_1 \to X, i = 1, 2$ are line-equivalent. Then $\rho(\pi_1) = \rho(\pi_2)$.*

3. Toric resolution and the computation of $\rho(\Sigma^*)$

3.1. **Non-degenerate surfaces.** We begin with recalling the toric resolutions of surface singularities since this also helps us to fix some notations. We use the notations of [9]. Let (X, O) be the germ of a surface in $(\mathbb{C}^3, O)$ defined by a function $f : (\mathbb{C}^3, O) \to (\mathbb{C}, O)$.

Hereafter we always assume that X has an isolated singularity at O. Let $\sum_{\nu} a_\nu z^\nu$ be the Taylor expansion of f. The *Newton polyhedron* $\Gamma_+(f)$ is by definition the convex hull of $\bigcup_{\{\nu;a_\nu\neq 0\}}\{\nu+\mathbb{R}^3\}$. The *Newton boundary* $\Gamma(f)$ is by definition the union of the compact faces of $\Gamma_+(f)$.

Let $N := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^3,\mathbb{Z})$ be the set of covectors. We identify N with $\mathbb{Z}^3$ and we denote the elements of N by column vectors. Let N_+ be the set of covectors $P = {}^{\mathrm{t}}(p_1,p_2,p_3)\in N$ with $p_i\geq 0, i=1,2,3$. Put $E_1 := {}^{\mathrm{t}}(1,0,0), E_2 := {}^{\mathrm{t}}(0,1,0)$ and $E_3 := {}^{\mathrm{t}}(0,0,1)$. P is called *strictly positive covector* if $p_j>0$ for all j. We denote the minimal value of the linear function P on $\Gamma_+(f)$ by $d(P;f)$. Put $\Delta(P;f)=\{z\in\Gamma_+(f)\mid P(z)=d(P;f)\}$. The *face function* of f with respect to P is by definition $f_P(z)=f_{\Delta(P;f)} := \sum_{\nu\in\Delta(P;f)} a_\nu z^\nu$. Two covectors $P,P'\in N_+$ are equivalent if and only if $\Delta(P;f)=\Delta(P';f)$. The *dual Newton diagram* $\Gamma^*(f)$ of X is a conical polyhedral subdivision of N_+ given by the above equivalent classes.

A surface X is called *non-degenerate* (with respect to the local coordinate z) if for any strictly positive covector $P\in N_+$, $X^*(P) := \{z\in\mathbb{C}^{*3}\mid f_P(z)=0\}$ is a reduced non-singular surface in the complex torus $\mathbb{C}^{*3}$. The notion of non-degeneracy can be extended to complete intersection varieties (cf. [6, 9]).

3.2. **Canonical subdivisions.** We assume that X is defined by $f(z_1,z_2,z_3)=0$ and f is non-degenerate. Let $\Gamma^*(f)_2^+$ be the union of the two-dimensional cones $\mathrm{Cone}(P,Q)$ of $\Gamma^*(f)$ such that the interior points are strictly positive. Let Σ^* be a regular simplicial subdivision of the dual Newton diagram $\Gamma^*(f)$ and let $\pi : X_{\Sigma^*}\to X$ be the associated toric modification. Let $\mathcal{V}(\Sigma^*)$ be the set of strictly positive vertices P's of Σ^* such that $\dim\Delta(P;f)\geq 1$. The exceptional divisors correspond bijectively to $\mathcal{V}(\Sigma^*)$ and for each $P\in\mathcal{V}(\Sigma^*)$ we denote the corresponding divisor by $E(P)$. Note that $E(P)$ need not to be irreducible but it is a disjoint union of rational spheres if $\dim\Delta(P;f)=1$. The number of connected components is given by $r(P)+1$, where $r(P)$ is the number of integral points on the interior of $\Delta(P;f)$ ([9, III§6]). The structure of this resolution $\pi : X_{\Sigma^*}\to X$ depends only on the restriction of Σ^* to $\Gamma^*(f)_2^+$. This follows from the following observation:

Proposition 4. *Assume that Σ_1^* is a regular subdivision of Σ^* such that $\mathcal{V}(\Sigma_1^*)=\mathcal{V}(\Sigma^*)$. Then the canonical morphism $\psi : X_{\Sigma_1^*}\to X_{\Sigma^*}$, which is induced by the morphism of the ambient toric varieties, is an isomorphism.*

For any two dimensional cone $\sigma=\mathrm{Cone}\,(P,Q)\in\Gamma^*(f)$, there exists a canonical regular subdivision of σ which is described as follows. Denote by $d := \det(P,Q)$ the greatest common divisor of the absolute values of the 2×2 minors of the matrix (P,Q). If $d>1$, there exists a unique integer $d_1, 1\leq d_1<d$ such that $Q_1 := (P+d_1Q)/d$ is an integral covector. If $d_1>1$, repeat the process for $\mathrm{Cone}\,(P,Q_1)$, until a regular subdivision of $\mathrm{Cone}\,(P,Q)$ is obtained. Let $Q_1,\dots,Q_k$ be the covectors obtained in this way. Let $d/d_1=[m_1,\dots,m_\ell]$ be the continuous fraction expansion. Then $\ell=k$ and the self-intersection number of each component of $E(Q_i)$ is $-m_i$ (cf. [9, III]). Note that $\Delta(Q_i;f)=\Delta(P;f)\cap\Delta(Q;f)$. This implies $r(Q_i)$ is independent of $i=1,\dots,k$ and we denote this number by $r(P,Q)$. Recall that the continuous fraction is defined inductively by $[m_1]=m_1$ and $[m_1,m_2,\dots,m_k]=m_1-1/[m_2,\dots,m_k]$.

A regular simplicial cone subdivision of $\Gamma^*(f)$ is called a *canonical regular subdivision* if its restriction to each cone σ in $\Gamma^*(f)_2^+$ is canonical in the above sense, and we denote it by $\Sigma^*_{\rm can}$. The associated toric resolution is called the *canonical toric resolution* of X.

Let $Q = {}^t(q_1, q_2, q_3)$ and $P = {}^t(p_1, p_2, p_3)$. Put $Q_0 = Q$ and $Q_{k+1} = P$ and let $Q_j := {}^t(q_{1,j}, q_{2,j}, q_{3,j}), j = 0, \dots, k+1$. The canonical subdivision enjoys the following property:

Lemma 5. *Assume that* $\mathrm{Cone}\,(P, Q) \in \Gamma^*(f)_2^+$. *Fix an* $\ell = 1, 2, 3$.

1) *If* $q_\ell \le 1$, *then* $\{q_{\ell,j}\}_{j=0}^{k+1}$ *is monotone increasing in* j *i.e.* $q_{\ell,j+1} \ge q_{\ell,j}$ *for* $0 \le j \le k$.
2) *If* $q_\ell \ge 2$, *then either* $\{q_{\ell,j}\}$ *is monotone increasing or monotone decreasing in* j *or there exists a* j_0 $(1 \le j_0 \le k)$ *such that* $q_{\ell,j_0} \ge 1$ *and*

$$p_\ell = q_{\ell,k+1} \ge \cdots \ge q_{\ell,j_0+1} \ge q_{\ell,j_0} \le q_{\ell,j_0-1} \le \cdots \le q_{\ell,0} = q_\ell.$$

Proof. We prove the assertion 2). If the assertion does not hold, there exists an index $j, 1 \le j \le k$ such that $q_{\ell,j-1} \le q_{\ell,j} > q_{\ell,j+1}$. This implies that the self intersection number of each component of $E(Q_j)$ is $-(q_{\ell,j-1} + q_{\ell,j+1})/q_{\ell,j} > -2$, which is a contradiction (cf. [9, II(2.3) and III(6.3)]). The assertion 1) follows from 2) as $Q_j, j = 1, \dots, k$ are strictly positive. $\square$

Let Σ^* be any regular simplicial cone subdivision of $\Gamma^*(f)$ and let $\pi : \tilde{X} \to X$ be the corresponding toric modification. We denote the line index of π by $\rho(\Sigma^*)$. Take a two dimensional cone $\sigma = \mathrm{Cone}(P, Q) \in \Gamma^*(f)_2^+$. Let $Q_0 := Q, Q_1, \dots, Q_k, Q_{k+1} := P$ be the canonical subdivision of σ and let $S_0 := Q, S_1, \dots, S_\eta, S_{\eta+1} := P$ be the vertices of Σ^* on this cone. By [9, II(2.3)], $\{Q_0, \dots, Q_{k+1}\} \subset \{S_0, \dots, S_{\eta+1}\}$. We consider the condition:

($\sharp$): Σ^* has no vertex in the interior of $\mathrm{Cone}(Q, Q_1)$.

We say that Σ^* satisfies the ($\sharp$)-condition if it satisfies ($\sharp$)-condition for any $\mathrm{Cone}\,(P, Q)$ in $\Gamma^*(f)_2^+$ such that Q is not strictly positive. The inclusion $\mathcal{V}(\Sigma^*_{\rm can}) \subset \mathcal{V}(\Sigma^*)$ implies that the following statements.

Theorem 6. *There exists a canonical morphism* $\phi : X_{\Sigma^*} \to X_{\Sigma^*_{\rm can}}$. *Furthermore* ϕ *is a composition of line-admissible blowing-ups if* Σ^* *satisfies* ($\sharp$)*-condition. In particular, the line index* $\rho(\Sigma^*)$ *does not depend on the choice of a toric resolution associated with any regular simplicial subdivision satisfying* ($\sharp$)*-condition and* $\rho(\Sigma^*) = \rho(\Sigma^*_{\rm can})$.

Proof. Take a two dimensional cone $\sigma = \mathrm{Cone}(P, Q) \in \Gamma^*(f)_2^+$ and assume that P is strictly positive. Let $Q_0 := Q, Q_1, \dots, Q_k, Q_{k+1} := P$ be the canonical subdivision of σ and let $S_0 := Q, S_1, \dots, S_\eta, S_{\eta+1} := P$ be the vertices of Σ^* on this cone. Write $S_i = {}^t(s_{1,j}, s_{2,j}, s_{3,j})$. Assume that $Q_{i_0} = S_\nu$ and $Q_{i_0+1} = S_\mu$ and $\mu - \nu > 1$. Take S_j with $\nu < j < \mu$ and put $\alpha_j = \det(Q_{i_0}, S_j)$ and $\beta_j = \det(S_j, Q_{i_0+1})$. Then α_j and β_j are positive integers and $S_j = \alpha_j Q_{i_0+1} + \beta_j Q_{i_0}$. This implies that $s_{1,j} > s_{1,\nu} + s_{1,\mu}$. Suppose that $s_1^{\max} = \max\{s_{1,j}; \nu < j < \mu\}$ and put $\gamma = \min\{\gamma; s_{1,\gamma} = s_1^{\max}\}$. Then by [9, II(2.3)] the intersection number of (each component of) $E(S_\gamma)$ is $-(s_{1,\gamma-1} + s_{1,\gamma+1})/s_{1,\gamma} > -2$. Then the negativity of the intersection number implies that $s_{1,\gamma-1} + s_{1,\gamma+1} = s_{1,\gamma}$. Thus each component of $E(S_\gamma)$ is a rational sphere of the first kind. This implies also that $S_\gamma = S_{\gamma-1} + S_{\gamma+1}$ and $\det(S_{\gamma-1}, S_{\gamma+1}) = 1$. Put $\mathcal{V}' = \mathcal{V}(\Sigma^*) - \{S_\gamma\}$. Then we can extend $\mathcal{V}'$ to get a regular simplicial subdivision $\Sigma^{*\prime}$

such that its restriction to $\Gamma^*(f)_2^+$ is defined by the vertices $\mathcal{V}'$. Thus we get a toric resolution $\pi' : X_{\Sigma^{*\prime}} \to X$. Changing Σ^* outside of $\Gamma^*(f)_2^+$ if necessary, we may assume by Proposition 4 that Σ^* is a subdivision of $\Sigma^{*\prime}$. Thus we get a canonical morphism $\psi : X_{\Sigma^*} \to X_{\Sigma^{*\prime}}$ which factors π by π'. By the definition, ψ is the composition of blowing-up at $r(S_\gamma)+1$ intersection points of respective components of $E(S_{\gamma-1})$ and $E(S_{\gamma+1})$ in $X_{\Sigma^{*\prime}}$. Note that ψ is line-admissible unless Q is not strictly positive and $S_\nu = Q_0$ and $S_\mu = Q_1$. This is the situation where ψ is the blowing up at the intersection of $E(Q_1)$ and $E(Q)$. This does not occur if Σ^* satisfies ($\sharp$)-condition. Now the assertion follows by the induction on the cardinality of $\mathcal{V}(\Sigma^*) \setminus \mathcal{V}(\Sigma^*_{\text{can}})$. □

3.3. **Computation of $\rho(\Sigma^*_{\text{can}})$.** Let $\pi : X_{\Sigma^*} \to X$ be a toric resolution. We assume that Σ^* satisfies the ($\sharp$)-condition. We define $\mathcal{V}_{\text{ns}}(\Sigma^*) := \{P \in \mathcal{V}(\Sigma^*) \mid P \text{ has 1 as a coordinate } \}$. We know that $E(P)$ is a normally smooth divisor if and only if $P \in \mathcal{V}_{\text{ns}}(\Sigma^*)$. Thus for each $\text{Cone}(P,Q) \in \Gamma^*(f)_2^+$, we define $\rho_{PQ} := \#\mathcal{V}_{\text{ns}}(\Sigma^*) \cap \text{Cone}\,(P,Q)^\circ$, where $\text{Cone}\,(P,Q)^\circ$ is the interior of $\text{Cone}\,(P,Q)$. This number is independent of Σ^* by Theorem 6. Recall that $r(P,Q)$ is the number of integral points in the interior of $\Delta(P;f) \cap \Delta(Q;f)$. By the definition we have

$$(1)\qquad \rho(\Sigma^*) = \sharp\{P \in \mathcal{V}_{\text{ns}}(\Sigma^*); \dim \Delta(P;f) = 2\} + \sum_{\text{Cone}\,(P,Q) \in \Gamma^*(f)_2^+} (r(P,Q)+1)\rho_{PQ}$$

Thus we need only to compute ρ_{PQ} for the calculation of $\rho(\Sigma^*)$. Take a cone $\sigma = \text{Cone}\,(P,Q)$ in $\Gamma^*(f)_2^+$. The following gives a practical method to compute ρ_{PQ}.

Theorem 7. *Let $P = {}^t(p_1,p_2,p_3)$ be strictly positive and let $Q = {}^t(q_1,q_2,q_3)$ and assume that $d := \det(P,Q) > 1$. Let $Q_i = {}^t(q_{1,i},q_{2,i},q_{3,i}), i = 0,\dots,k+1$ be the vertices defining the canonical subdivision from Q with $Q_0 = Q$ and $Q_{k+1} = P$. Fix an $\ell \in \{1,2,3\}$. Then*

1. *For each $1 \le i \le k$, there exists positive integers $0 < \alpha_i, \beta_i < d$ such that $Q_i = (\beta_i P + \alpha_i Q)/d$. Putting $\alpha_0 = \beta_{k+1} = d$, $\alpha_{k+1} = \beta_0 = 0$, they satisfy the inequality:*
$$\alpha_i > \alpha_{i+1}, \quad \beta_i < \beta_{i+1}, \quad i = 0,\dots,k$$
2. *Let $\mathcal{V}_{\text{ns}}^{(\ell)}(P,Q)$ be the set of integral covectors R expressed as $R = (\beta P + \alpha Q)/d$ where α, β are positive integers satisfying*
$$(2)\qquad \begin{cases} \alpha q_\ell + \beta p_\ell = d, \ 0 < \alpha, \beta < d \\ \alpha q_k + \beta p_k \equiv 0 \mod d \quad (k \ne \ell) \end{cases}$$
and let $\mathcal{V}_{\text{ns}}^{(\ell)}(P,Q;\Sigma^*_{\text{can}})$ be the set of covectors Q_i, $1 \le i \le k$ such that $q_{\ell,i} = 1$. Then $\mathcal{V}_{\text{ns}}^{(\ell)}(P,Q) = \mathcal{V}_{\text{ns}}^{(\ell)}(P,Q;\Sigma^*_{\text{can}})$. Note that the inequality $\alpha,\beta < d$ follows automatically from the positivity if both p_ℓ and q_ℓ are positive.

Proof. The first assertion follows by an inductive argument. Write $Q_i = (\beta_i P + \alpha_i Q)/d$ with positive rational numbers α_i, β_i. As $\det(P,Q_i) = \alpha_i$ and $\det(Q_i,Q) = \beta_i$, α_i, β_i are positive integers. By the definition of Q_1, we can write $Q_1 = (P + \alpha_1 Q)/d$ for some $0 < \alpha_1 < d$.

The assertion for Q_1 holds and $\det(P,Q_1)=\alpha_1$. Assume that $Q_j=(\beta_jP+\alpha_jQ)/d$ with $0<\alpha_j<d$. As $\det(P,Q_j)=\alpha_j$ and $\{Q_j,\dots,Q_{k+1}\}$ is the vertices of the canonical subdivision of Cone (P,Q_j), there exists α', $0<\alpha'<\alpha_i$, such that

$$Q_{j+1}=\frac{1}{\alpha_j}P+\frac{\alpha'}{\alpha_j}Q_j=\frac{1}{\alpha_j}P+\frac{\alpha'}{\alpha_j}\frac{(\beta_jP+\alpha_jQ)}{d}=(\frac{1}{\alpha_j}+\frac{\alpha'\beta_j}{\alpha_jd})P+\frac{\alpha'}{d}Q$$

Thus $\alpha_{j+1}=\alpha'<\alpha_j<d$. The inequality $\beta_{j+1}>\beta_j$ can be proved similarly by using the fact that $\{P,Q_k,\dots,Q_1,Q\}$ is the vertices of the canonical subdivision of the cone Cone (P,Q) from P (cf. [9, II(2.3)]). Now we show the second assertion. The inclusion $\mathcal{V}_{\rm ns}^{(\ell)}(P,Q;\Sigma_{\rm can}^*)\subset\mathcal{V}_{\rm ns}^{(\ell)}(P,Q)$ is obvious. Suppose that $R=(\beta P+\alpha Q)/d\in\mathcal{V}_{\rm ns}^{(\ell)}(P,Q)$ is not contained in $\mathcal{V}_{\rm ns}^{(\ell)}(P,Q;\Sigma_{\rm can}^*)$. Suppose that $R\in\mathrm{Cone}\,(Q_i,Q_{i+1})^\circ$. Then we can write $R=mQ_i+nQ_{i+1}$ for some positive integers m,n. If $i\ge 1$, this gives a contradiction by comparing the ℓ-th coefficient: $1=mq_{\ell,i}+nq_{\ell,i+1}\ge m+n$. Suppose that $i=0$. Write $Q_1=(P+\alpha_1Q)/d$ as above. Then $R=mQ+(P+\alpha_1Q)n/d=nP/d+(md+n\alpha_1)Q/d$. Thus we get $\alpha=md+\alpha_1n\ge d$ which contradicts to the assumption. □

Remark 8. The computation of $\mathcal{V}_{\rm ns}(P,Q)$ is most difficult for the case $p_\ell,q_\ell>1$. Assume that $p_\ell,q_\ell>0$. If we have a solution (α_0,β_0), the other solutions are reduce to the following equation. Put $\alpha=\alpha_0+\alpha',\beta=\beta_0+\beta'$. Then

$$\left\{\begin{array}{l}\alpha'q_\ell+\beta'p_\ell=0\\ \alpha'q_k+\beta'p_k\equiv 0\quad\mathrm{mod}\ d\quad(k\ne\ell)\end{array}\right.\tag{3}$$

Let $\Delta:=\Delta(P;f)\cap\Delta(Q;f)$. Let $T={}^{\rm t}(t_1,t_2,t_3)$ be a covector in $\mathcal{V}_{\rm ns}^{(\ell)}(P,Q)$ (thus $t_\ell=1$). Geometrically this implies that $\Delta(T;f)=\Delta$. In particular, $\Gamma_+(f)\subset\{(\nu_1,\nu_2,\nu_3);t_1\nu_1+t_2\nu_2+t_3\nu_3\ge d(T;f)\}$. This gives a practical way to find such a T.

The case $q_\ell=0$ or 1, the computation is much easier. See Corollary 11.

The canonical subdivision of Cone (P,Q) takes sometimes a lot of computations (see Example 9). Theorem 7 gives us a criterion on the existence or non-existence of normally smooth divisors, without computing the whole subdivision $Q_i,i=1,\dots,k$.

Example 9. For simplicity, we write $x=z_1,y=z_2,z=z_3$. Let us consider $f(x,y,z)=x^m+y^n+x^ry^r+z^2$. We assume that $m,n>2r$. Put $n=n_1r+n_0,m=m_1r+m_0$ with $0\le m_0,n_0\le r-1$. Then $\Gamma(f)$ has two compact faces whose covectors are $P={}^{\rm t}(2(n-r),2r,nr)/\delta_1$ and $Q={}^{\rm t}(2r,2(m-r),mr)/\delta_2$ where $\delta_1=\gcd(2(n-r),2r,nr)$ and $\delta_2=\gcd(2r,2(m-r),mr)$ and the corresponding dual Newton diagram is as in Figure 1. Note that $d:=\det(P,Q)$ is given by $d=2(mn-mr-nr)/(\delta_1\delta_2)$. We consider $\mathcal{V}_{\rm ns}^{(1)}(P,Q)$. First we consider the covector $T_0={}^{\rm t}(1,1,r)$, which is a weight vector of $x^ry^r+z^2$. As $m,n>2r$, T_0 must be on Cone (P,Q). To proceed the further computation, let us assume that n,m,r are odd and

$\gcd(m, r) = \gcd(n, r) = 1$. This implies $\delta_1 = \delta_2 = 1$. By Theorem 7, we have

$$\begin{cases} 2\beta(n-r) + 2\alpha r = d \\ 2\beta r + 2\alpha(m-r) \equiv 0 \mod d \\ \beta nr + \alpha mr \equiv 0 \mod d \end{cases}$$

First we have a canonical solution $(\alpha_0, \beta_0) = (n-2r, m-2r)$ which corresponds to the covector $T_0 = {}^t(1, 1, r)$. Thus putting $\alpha = \alpha_0 + a$ and $\beta = \beta_0 + b$, we can reduce the equation as

$$\begin{cases} 2b(n-r) + 2ar = 0 \\ 2br + 2a(m-r) \equiv 0 \mod d \\ bnr + amr \equiv 0 \mod d \end{cases}$$

Taking the positivity of α, β into account, we have the solution

$$\{(\alpha, \beta)\} = \left\{((n-2r) + 2j(n-r), (m-2r) - 2jr)\,; 0 \le j \le \left[\frac{m_1 - 2}{2}\right]\right\}$$

For example, consider the easiest case $m = n$. This has a unique solution $(\alpha, \beta) = (n - 2r, n-2r)$ and $\mathcal{V}_{\rm ns}^{(1)}(P, Q) = \{B\}$ where $B = {}^t(1, 1, r)$. By symmetry, we have $\mathcal{V}_{\rm ns}^{(2)} = \{B\}$. Note $r(P, Q) = 1$. By writing down the equation described by Theorem 7, we can show $\mathcal{V}_{\rm ns}^{(3)}(P, Q) = \emptyset$.

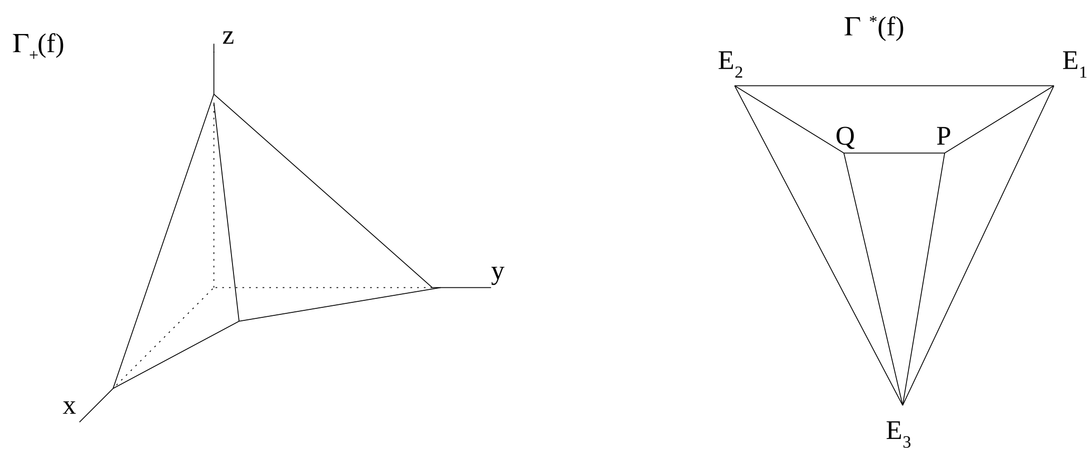

FIGURE 1. The Newton polyhedron and the dual Newton diagram

Now we look at Cone (P, E_1) and Cone (P, E_3). Note that $\det(P, E_1) = r$ and $\det(P, E_3) = 2$. It is easy to see that there are no normally smooth divisor on these cones. Observe that the computation of canonical subdivision of Cone (P, Q) is not so easy. For example, if $r = 15, n = 37$, then $B = {}^t(1, 1, 15)$ and first covector B_1 (from Q) is given by $(P + 223Q)/518 = {}^t(13, 19, 240)$ and $518/223 = [3, 2, 2, 12, 2, 2, 3]$ and it takes some computation to complete the subdivision.

The following lemma describes the covectors corresponding to the non-compact faces.

Lemma 10. *Assume that* $X = \{f(z_1, z_2, z_3) = 0\}$ *and assume that* f *is non-degenerate and* $\Gamma(f)$ *has at least one compact two dimensional face for simplicity. Suppose that* $z_2 = z_3 = 0$ *is a line in* X*. (So* f *is not convenient.) Then there is a unique covector* $Q = {}^t(q_1, q_2, q_3) \in \mathrm{Vertex}(\Gamma^*(\mathrm{f}))$ *such that* $q_1 = 0$*. Furthermore* Q *takes the form* ${}^t(0, 1, q_3)$ *or* ${}^t(0, q_2, 1)$*.*

There exists a unique covector $P = {}^t(p_1, p_2, p_3)$ *which corresponds to a compact divisor and adjacent to* Q *in* $\Gamma^*(f)_2^+$*. Then we have* $\det(P, Q) = p_1$*.*

Proof. As X has an isolated singularity, f must contain a monomial of type $z_1^a z_2$ or $z_1^a z_3$. Suppose that $B := (a, 1, 0) \in \Gamma(f)$. Let $C = (b, 0, c)$ be the vertex of $\Gamma(f) \cap \{z_2 = 0\}$ adjacent to B by an edge. It is clear that the non-compact face Ξ which has $\overline{BC}$ as a face and is unbounded to the direction of the z_1-axis has covector $Q = {}^t(0, c, 1)$. One can see that there exists no other non-compact face which is unbounded to the z_1-axis direction and bounded to z_2, z_3-direction. Let Δ be the compact face which has $\overline{BC}$ as a boundary and let $P = {}^t(p_1, p_2, p_3)$ be the corresponding covector. As $\Delta(P; f)$ contains B, C, we need to have $p_1 a + p_2 = bp_1 + cp_3$. Now the last assertion follows from $\det(P, Q) = \gcd(p_1, p_2 - cp_3) = \gcd(p_1, p_1(b - a)) = p_1$. □

The following corollary describes explicitly $\mathcal{V}_{\mathrm{ns}}^{(1)}(P, Q)$ in the case $q_1 = 0$ or 1.

Corollary 11. *With the assumptions of Theorem 7, we have the following.*

1) *Assume* $q_1 = 0$*. Then* $\mathcal{V}_{\mathrm{ns}}^{(1)}(P, Q) \neq \emptyset$ *if and only if* $d := \det(P, Q) > 1$ *and* $d = p_1$*. In this cases,* $\mathcal{V}_{\mathrm{ns}}^{(1)}(P, Q) = \{Q_1\}$*. If* $Q \neq E_2, E_3$*, then* $\{y = z = 0\} \subset X$ *and* $d = \det(P, Q) = p_1$*.*
2) *Assume* $q_1 = 1$*. Then* $\mathcal{V}_{\mathrm{ns}}^{(1)}(P, Q) \neq \emptyset$ *if and only if* $d > p_1$*. In this case, we have* $Q_i = (iP + (d - ip_1)Q)/d$ *for* $i = 1, \dots, [d/p_1]$ *and* $\mathcal{V}_{\mathrm{ns}}^{(1)}(P, Q) = \{Q_i; i = 1, \dots, [d/p_1]\}$*.*

Proof. Assume that $Q' = (\beta P + \alpha Q)/d \in \mathcal{V}_{\mathrm{ns}}^{(1)}(P, Q)$ with $0 < \alpha, \beta < d$.

1) If $q_1 = 0$, we have $\gcd(q_2, q_3) = 1$. As $d = \gcd(p_1 q_2, p_1 q_3, p_2 q_3 - p_3 q_2) = \gcd(p_1, p_2 q_3 - p_3 q_2)$, d divides p_1. Thus $Q' \in \mathcal{V}_{\mathrm{ns}}^{(1)}(P, Q)$ if and only if $d = p_1$ and $\beta = 1$. In this case, $Q' = Q_1$ and $\mathcal{V}_{\mathrm{ns}}^{(1}(P, Q) = \{Q_1\}$. Assume that $Q \neq E_2, E_3$. By the definition of $\Gamma^*(f)_2^+$, $\Delta(Q; f)$ is a non-compact face with dimension 2. In particular, $\{y = z = 0\} \subset X$. By Lemma 10, we have $d = p_1$.

2) Suppose that $q_1 = 1$. Then $\beta p_1 + \alpha = d$. This implies $d > p_1$. Put $d = rp_1 + d'$ with $0 \le d' < p_1$ and $r = [d/p_1]$. Then by the above equality, we have $(\alpha, \beta) = (d - jp_1, j)$, $j = 1, \dots, [d/p_1]$. Put $Q'_j := (jP + (d - jp_1)Q)/d$. By the definition, d divides the minors of (P, Q) which are $p_1 q_2 - p_2, p_1 q_3 - p_3, p_2 q_3 - p_3 q_2$. Thus $\beta p_j + \alpha q_j = \beta p_j + (d - \beta p_1) q_j \equiv \beta(p_j - p_1 q_j) \equiv 0 \mod d$ for $j = 2, 3$. Thus Q'_j is an integral covector for $\beta = 1, \dots, r$. It is clear that $Q'_1 = Q_1$. Assume that $Q'_r = Q_\iota$ for some ι. By the monotonity of the coefficients (Lemma 5), we have $Q_j \in \mathcal{V}_{\mathrm{ns}}^{(1)}(P, Q)$ for $j \le \iota$. Thus $\iota = r$ and $Q'_j = Q_j$ for $j \le r$. □

Remark 12. In the case of non-convenient surface with $q_1 = 0$, the divisor $E(Q_1)$ corresponds to the deformations of the line $z_2 = z_3 = 0$. In fact, $E(Q)$ is a non-compact divisor which is the strict transform of z_1-axis and $E(Q)$ intersects transversely with $E(Q_1)$.

For $R \in \mathcal{V}_{\mathrm{ns}}^{(\ell)}$, write $R = (\beta P + \alpha Q)/d$. We call β/d *the* P*-coefficient* of R.

Corollary 13. *With the assumptions of Theorem 7, suppose that* $q_1 > 1$. *Let* $\bar{Q} = (\bar{\beta}P + \bar{\alpha}Q)/d \in \mathcal{V}_{\rm ns}^{(\ell)}$ *and* $\underline{Q} = (\underline{\beta}P + \underline{\alpha}Q)/d \in \mathcal{V}_{\rm ns}^{(\ell)}$ *be the covectors with maximal and minimal* P*-coefficients in* $\mathcal{V}_{\rm ns}^{(\ell)}$. *Then*

$$\rho_{PQ}^{(\ell)} = 1 + |\det(\bar{Q}, \underline{Q})| = 1 + \frac{|\bar{\beta}\underline{\alpha} - \bar{\alpha}\underline{\beta}|}{d} \tag{4}$$

Proof. Denote by $d' := |\det(\bar{Q}, \underline{Q})|$. Suppose that $\underline{Q} = Q_i$ and $\bar{Q} = Q_{i+j}$. Then $\mathcal{V}_{\rm ns}^{(\ell)} = \{Q_i, \dots, Q_{i+j}\}$ by Lemma 5 and $\rho_{PQ}^{(\ell)} = j + 1$. By the assumption, we have $Q_{i+1} = (Q_{i+j} + (d'-1)Q_i)/d'$. As the continuous fraction $d'/(d'-1)$ is given by $[2, \dots, 2]$ ($(d'-1)$ copies of 2), we get $j - 1 = d' - 1$ and the assertion follows immediately. □

4. Applications

4.1. **Weighted homogeneous surfaces.** In this section we study lines on weighted homogeneous surface singularities, which are classified as follows ([12, 9]):

$X_{\rm I}$: $h_{\rm I} = x^a + y^b + z^c = 0,$

$X_{\rm II}$: $h_{\rm II} = x^a y + y^b + z^c = 0,$

$X_{\rm III}$: $h_{\rm III} = x^a y + x y^b + z^c = 0,$

$X_{\rm IV}$: $h_{\rm IV} = x^a y + y^b z + z^c = 0,$

$X_{\rm V}$: $h_{\rm V} = x^a y + y^b z + z^c x = 0,$

$X_{\rm VI}$: $h_{\rm VI} = xy + z^c = 0,$

$X_{\rm VII}$: $h_{\rm VII} = x^a z + y^b z + z^c + t x^{c_1} y^{c_2} = 0, \quad t \neq 0$

$X_{\rm VIII}$: $h_{\rm VIII} = x^a y + x y^b + x z^c + t y^{c_1} z^{c_2} = 0, \quad t \neq 0.$

The surface $X_{\rm I}$ is called a Pham-Brieskorn surface. This type of surfaces have been studied in the previous paper [4]. The surface $X_{\rm VI}$ is an A_{c-1} type singularity. There are exact $c - 1$ families of lines on this surface (see [1, 2, 4, 5]). On surface $X_{\rm VII}$ and $X_{\rm VIII}$, the term $y^{c_1} z^{c_2}$ must be on the supporting plane of the previous three monomials. Thus a, b, c are not arbitrary. The Newton boundaries of the surfaces other than $X_{\rm VI}, X_{\rm VII}$ and $X_{\rm VIII}$ are triangles. Note that for a weighted homogeneous surface, the Newton boundary has only one compact 2-dimensional face. Let $P = {}^t(p_1, p_2, p_3)$ be the corresponding covector. The formula (1) in §2 reduces to

$$\rho(\Sigma_{\rm can}^*) = \varepsilon + \sum_{{\rm Cone}\,(P,Q) \in \Gamma^*(f)_2^+} (r(P,Q) + 1)\rho_{PQ}(\Sigma_{\rm can}^*). \tag{5}$$

where $\varepsilon = 1$ if $P \in \mathcal{V}_{\rm ns}(\Sigma_{\rm can}^*)$ and $\varepsilon = 0$ otherwise.

For each type of surfaces, one can calculate $\rho_{PQ}(\Sigma_{\rm can}^*)$ for each ${\rm Cone}\,(P,Q)$ in the dual Newton diagram by using the method described in the previous sections.

Lemma 14. *Assume that* ${\rm Cone}\,(P, E_i)$ *be a cone in* $\Gamma^*(f)_2^+$. *Then* $\det(P, E_i)$ *is given by* $\delta_i := \gcd(p_j, p_k)$ *with* $\{i, j, k\} = \{1, 2, 3\}$. *Assume that* $\delta_i > 1$.

1) $\mathcal{V}_{\rm ns}^{(i)}(P, E_i) \neq \emptyset$ *if and only if* $\delta_i > p_i$ *and* $\rho_{PE_i}^{(i)} = \left[\frac{\delta_i}{p_i}\right]$.
2) $\mathcal{V}_{\rm ns}^{(j)}(P, E_i) \neq \emptyset$ *if and only if* $p_j | p_k$. *In this case,* $\rho_{PE_i}^{(j)} = 1$.

3)

$$\rho_{PE_i} = \begin{cases} 0, \quad \textit{if } \left[\frac{\delta_i}{p_i}\right] = 0 \textit{ and } \delta_i < \min\{p_j, p_k\} \\ \max\{1, \left[\frac{\delta_i}{p_i}\right]\}, \quad \textit{otherwise} \end{cases}$$

Proof. This follows from Corollary 11. □

Lemma 15. *Let* Cone (P, Q) *be a cone in* $\Gamma^*(f)_2^+$ *with* $Q = {}^t(0, c, 1)$*. Suppose that* $\det(P, Q) = p_1 > 1$*. Then*

$$\rho_{PQ} = \begin{cases} \max\{1, \left[\frac{p_1}{p_2}\right], \left[\frac{p_1}{p_3}\right]\}, \quad c = 1 \\ \rho_{PQ}^{(2)} + \max\{1, \left[\frac{p_1}{p_3}\right]\} - \varepsilon, \quad c > 1 \end{cases}$$

where $\varepsilon = 1$ *if either* $Q_1 \in \mathcal{V}_{\mathrm{ns}}^{(2)}(P, Q)$ *or* $Q_{j_1} \in \mathcal{V}_{\mathrm{ns}}^{(2)}(P, Q)$ *with* $j_1 := \left[\frac{p_1}{p_3}\right] \geq 1$ *and* $\varepsilon = 0$ *otherwise.*

Proof. Let $Q_1, \dots, Q_k$ be the primitive covectors in Cone (P, Q) inserted by the canonical subdivision from Q. If $c = 1$, the assertion is immediate from Corollary 11, as $q_{1,1} = 1$. We assume that $c > 1$. If $[p_1/p_3] = 0$, the assertion is obvious. Assume that $[p_1/p_3] \geq 1$. By Corollary 11, Q_j is given by $(jP + (p_1 - jp_3)Q)/p_1$ for $1 \leq j \leq j_1$. Thus $q_{2,j} = c - j(cp_3 - p_2)/p_1$. If $cp_3 - p_2 < 0$, $q_{2,j}$ is monotone increasing by Lemma 5 and we see that $\mathcal{V}_{\mathrm{ns}}^{(2)}(P, Q) = \emptyset$ and the assertion follows immediately. Assume that $cp_3 - p_2 \geq 0$. Then $q_{2,j}$ is monotone decreasing for $0 \leq j \leq j_1$. Thus $\mathcal{V}_{\mathrm{ns}}^{(2)}(P, Q) \cap \mathcal{V}_{\mathrm{ns}}^{(3)}(P, Q) \neq \emptyset$ if and only if $q_{2,j_1} = 1$. If this is the case, Q_{j_1} is the unique covector in common. Thus the assertion follows from these observations. □

4.2. **Normally smooth divisors on X_{II}.** By using Lemmas 14 and 15, we can compute the number $\rho(\Sigma_{\mathrm{can}}^*)$. We show this by considering the surface X_{II}. One can do the same consideration for the other types of surfaces. Let $X_{\mathrm{II}} : h_{\mathrm{II}}(x, y, z) = x^a y + y^b + z^c = 0$. Put $\hat{a} := \gcd(a, b-1), e := \gcd(b, c)$ and $d := \gcd(c(b-1), ac, ab) = e \gcd(a, c(b-1)/e)$. The dual Newton diagram $\Gamma^*(h_{\mathrm{II}})_2^+$ consists of three cones: Cone (P, Q), Cone (P, E_1) and Cone (P, E_3) where $P := {}^t(c(b-1)/d, ac/d, ab/d)$ and $Q := {}^t(0, c, 1)$.

The following three propositions are special cases of Lemmas 14 and 15.

Proposition 16. Cone (P, E_1) *is regular if and only if a divides $c(b-1)/e$. Assume that* $a \nmid (c(b-1)/e)$*. Then*

1) $\mathcal{V}_{\mathrm{ns}}^{(1)}(P, E_1) \neq \emptyset$ *if and only if* $ae > (b-1)c$*. And in this case* $\rho_{PE_1}^{(1)} = \left[\frac{ae}{(b-1)c}\right]$.
2) $\mathcal{V}_{\mathrm{ns}}^{(2)}(P, E_1) \neq \emptyset$ *if and only if* $c|b$*.*
3) $\mathcal{V}_{\mathrm{ns}}^{(3)}(P, E_1) \neq \emptyset$ *if and only if* $b|c$*.*
4) $\rho_{PE_1} = \max\{\rho_{PE_1}^{(2)}, \rho_{PE_1}^{(3)}, \left[\frac{ae}{(b-1)c}\right]\}$. □

Proposition 17. *As* $\det(P, E_3) = c\hat{a}/d$, Cone (P, E_3) *is regular if and only if* $d = c\hat{a}$*. Assume that* $c\hat{a} > d$*. Then*

1) $\mathcal{V}_{\mathrm{ns}}^{(1)}(P, E_3) \neq \emptyset$ *if and only if* $(b-1)|a$*.*
2) $\mathcal{V}_{\mathrm{ns}}^{(2)}(P, E_3) \neq \emptyset$ *if and only if* $a|(b-1)$*.*

3) $\mathcal{V}_{\rm ns}^{(3)}(P,E_3)\neq\emptyset$ *if and only if* $c\hat a>ab$ *and* $\rho_{PE_3}^{(3)}=\left[\frac{c\hat a}{ab}\right]$.
4) $\rho_{PE_3}=\max\{\rho_{PE_3}^{(1)},\rho_{PE_3}^{(2)},\left[\frac{c\hat a}{ab}\right]\}$.

Recall that $\rho_{P,E_i}^{(j)}\le 1$ for $i=1,3$ and $j\neq i$ by Lemma 5.

Proposition 18. Cone$\,(P,Q)$ *is regular if and only if* $(b-1)c$ *divides* ae*, or equivalently* $(b-1)|a$ *and* $c|b\frac{a}{b-1}$*. Assume that* Cone$\,(P,Q)$ *is not regular. Then we have*

1) $\mathcal{V}_{\rm ns}^{(1)}(P,Q)=\{Q_1\}$.
2) $\mathcal{V}_{\rm ns}^{(3)}(P,Q)\neq\emptyset$ *if and only if* $c(b-1)>ab$*. And in this case* $\rho_{PQ}^{(3)}=\left[\frac{c(b-1)}{ab}\right]$.
3) $\mathcal{V}_{\rm ns}^{(2)}(P,Q)\neq\emptyset$ *if and only if there exist positive integers* α *and* β *such that*

$$a\beta+d\alpha=b-1, \tag{6}$$

$$ab\beta+d\alpha\equiv 0 \mod c(b-1). \tag{7}$$

The second condition can be replaced by $a\beta+1\equiv 0\quad$ modulo c.

Proof. The last assertion follows from by (6) as $ab\beta+d\alpha=(b-1)(a\beta+1)$. □

The non-trivial computation is required only for $\mathcal{V}_{\rm ns}^{(2)}(P,Q)$ which we will explain more in detail. Write $b=eb_1$ and $c=ec_1$.

Corollary 19. I. *For* $\mathcal{V}_{\rm ns}^{(2)}(P,Q)\neq\emptyset$*, it is necessary that*

$$\gcd(a,c)=1,\quad b>a,c \tag{8}$$

In this case, we have $d=e\hat a$ *and* $\mathcal{V}_{\rm ns}^{(2)}(P,Q)$ *is the set of covectors* $T=(\alpha Q+\beta P)/d$ *which satisfies*

$$a\beta+e\hat a\alpha=b-1 \tag{9}$$

$$0<\alpha,\beta \tag{10}$$

$$b-e\hat a\alpha\equiv 0 \text{ modulo c} \tag{11}$$

II. *Furthermore* $\mathcal{V}_{\rm ns}^{(2)}(P,Q)$ *is non-empty if* $[b/c]\ge a+\hat a$.

Proof. From the congruence $a\beta+1\equiv 0$ modulo c, it is clear that $\gcd(a,c)=1$. Thus $d=e\gcd(a,c_1(b-1))=e\hat a$. The equality (11) results from

$$a\beta+1=b-d\alpha=e(b_1-\hat a\alpha)\equiv 0 \text{ modulo c}$$

Thus $b>a\beta\ge a$ and $b>c$. The last congruence equation is equivalent to $b_1-\hat a\alpha\equiv 0$ modulo c_1.

Assume that $[b/c]-a-\hat a\ge 0$. As $\gcd(\hat a,b_1)=1$, there exists positive integer α_0, $0<\alpha_0<c_1$, such that $b_1-\hat a\alpha_0\equiv 0$ modulo c_1. Put $b_1-\alpha_0\hat a=j_0c_1$. We see that $j_0=b_1/c_1-\alpha_0\hat a/c_1>[b/c]-\hat a$. Take α which satisfies the congruence $a\beta+1\equiv 0$ modulo c. Then α takes the form $\alpha=\alpha_0+jc_1$ with $j\in\mathbf{N}$ and thus $b_1-\hat a\alpha=(j_0-j\hat a)c_1$. For the positivity of β, we need to have $0\le j<j_0/\hat a$. The integrity of T implies

$$e(b_1-\hat a\alpha)-1=ec_1(j_0-j\hat a)-1\equiv 0\quad\text{modulo a}$$

As j can move $0 \le j < j_0/\hat{a}$ and $j_0 > [b/c] - \hat{a} \ge a$ or $j_0/\hat{a} > a/\hat{a}$, this congruence equation has a positive solution j_1, $0 \le j_1 \le j_0/\hat{a}$. Then put $\beta = (ec_1(j_0 - j_1\hat{a}) - 1)/a$ for such a solution j_1. This gives a covector $T = (\alpha Q + \beta P) \in \mathcal{V}_{\mathrm{ns}}^{(2)}(P, Q)$. □

Example 20. Consider $X_{\mathrm{II}} : x^9 y + y^b + z^8 = 0$ with $b = 22 + 36k$. Then $e = 2, \hat{a} = 3$ and the equation is

$$9\beta + 6\alpha = 21 + 36k, \quad 9\beta + 1 \equiv 0 \text{ modulo } 8$$

In this case, $[b/c] - a - \hat{a} = (22 + 36k)/8 - 12 \ge 0$ if $k \ge 37/18$. For $k \ge 3$ (in fact, for $k \ge 2$), we have a solution $(\alpha, \beta) = (6k - 7, 7)$. In this case, $P = {}^{\mathrm{t}}(28 + 48k, 12, 33 + 54k)$ and $Q = {}^{\mathrm{t}}(0, 8, 1)$ and $T := (\alpha Q + \beta Q)/(28 + 48k) = {}^{\mathrm{t}}(7, 1, 8)$. We leave the computation of the other covectors in $\mathcal{V}_{\mathrm{ns}}^{(2)}(P, Q)$ to the reader.

4.3. **The minimality of the canonical toric resolutions.** We study when the canonical toric resolution of a weighted homogeneous surface is minimal. Though the canonical toric resolution is not always minimal (see Example 28), we can expect that the minimality hold for almost all classes of non-degenerate surfaces. By [9, III(6.3)], for each weighted homogeneous surface the resolution graph associated with the canonical toric resolution is star-shaped. Hence, when the resolution graph has at least three arms, the canonical resolution is minimal.

We have the following general statement which is very helpful to see if a given toric modification is minimal.

Lemma 21. *Let $X := f^{-1}(0)$ be a non-degenerate surface. Suppose that $P \in \Gamma^*(f)$ is the strictly positive covector corresponding to a compact face Δ of the Newton boundary $\Gamma(f)$.*

1) *Let $\Delta_1, \dots, \Delta_\ell$ be the boundary edges of Δ. The exceptional divisor $E(P)$ is rational if and only if*

$$-\frac{6\mathrm{Vol}(\mathrm{Cone}\,\Delta)}{d(P; f)} + \sum_{i=1}^{l} (r(\Delta_i) + 1) = 2$$

where $\mathrm{Cone}\,\Delta$ is the cone over Δ with vertex O and $r(\Delta_i)$ is the number of integral points in the interior of Δ_i.

2) *The canonical toric resolution $\pi : \tilde{X} \longrightarrow (X, 0)$ is not minimal if and only if there exists a compact face Δ of $\Gamma(f)$ such that $E(P)$ is rational, $E(P)^2 = -1$ and $E(P)$ intersects at most two other exceptional divisors where P is the covector corresponding to Δ.*

Proof. The first statement is a conclusion of [9, III(6.4)]. The assertion 2) follows from the Castelnuovo-Enriques criterion and [9, III §4(A) and §6]. □

Theorem 22. *Let X be one of the surfaces of type X_{II}, X_{III}, X_{IV}, X_{V}, X_{VII} or X_{VIII}. We assume that $a, b, c > 1$ in 4.1. Then the canonical toric resolution of X is minimal. In particular, $\rho(X, 0) = \rho(\Sigma^*_{\mathrm{can}})$.*

Proof. We first check when the central exceptional divisor $E(P)$ is rational by using Lemma 21 (see also [9, III(6.9)]). If this is the case, we compute the number of arms from $E(P)$. If this

number is less than 3, we show that $E(P)^2 \le -2$. Recall that the number of arms in the resolution graph is the sum of $r(P,Q)+1$ for non-regular cones $\mathrm{Cone}\,(P,Q) \in \Gamma^*(f)_2^+$.

(II). Let $X = X_{\mathrm{II}} : x^a y + y^b + z^c = 0$. Put $e = \gcd(b,c), \hat{a} = \gcd(a, b-1)$. Then $P = {}^t(c(b-1), ac, ab)/d$ with $d = e\gcd(a, c(b-1)/e)$. Note that $r(P,Q)+1 = 1$, $r(P,E_1)+1 = e$ and $r(P,E_3)+1 = \hat{a}$. By loc. cit. $E(P)$ is rational if and only if 1) $e = \gcd(c, a/\hat{a}) = 1$ or 2) $\hat{a} = \gcd(a, c/e) = 1$. If 1) holds, then $d = \hat{a}$. We have $\det(P,Q) = c(b-1)/\hat{a} > 1$, $\det(P,E_3) = c > 1$ and $\det(P,E_1) = a/\hat{a}$. If $\hat{a} = a$, $\mathrm{Cone}\,(P,E_3)$ gives $\hat{a} = a$ arms. Hence, in any case the resolution graph of X_{II} has at least three arms centered at $E(P)$.

In case 2), we have $\det(P,Q) = c(b-1)/e > 1$, $\det(P,E_1) = a > 1$ and $\det(P,E_3) = c/e$. If $e < c$, we have at least three arms in the resolution graph. Suppose that $e = c$. Then the number of arms at $E(P)$ is $e+1 \ge 3$, unless $b = 2$ and $e = c = 2$. In this case, the resolution graph has two similar arms and $E(P)$ is normally smooth with $E(P)^2 \le -2$.

Outline of other cases:

(III) Let $X_{\mathrm{III}} : x^a y + x y^b + z^c = 0$. Then $P = {}^t(c(b-1), c(a-1), ab-1)/d$ with $d = e\gcd(c, (ab-1)/e)$ and $e = \gcd(a-1, b-1)$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 3 arms $\mathrm{Cone}\,(P,E_3)$, $\mathrm{Cone}\,(P,Q)$, $\mathrm{Cone}\,(P,R)$ where $Q = {}^t(0,c,1)$ and $R = {}^t(c,0,1)$. The central divisor $E(P)$ is rational if and only if $\gcd(c,(ab-1)/e) = 1$. If $E(P)$ is rational, then $d = e$ and $\det(P,Q) = c(b-1)/e > 1, \det(P,R) = c(a-1)/e > 1$, and $\det(P,E_3) = c > 1$. Hence, the resolution graph has at least three arms.

(IV) Let $X_{\mathrm{IV}} : x^a y + y^b z + z^c = 0$. Then $P := {}^t(bc - c + 1, a(c-1), ab)/d$ with $d = e\gcd(a, (bc-c+1)/e)$ and $e := \gcd(b, c-1)$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 3 arms $\mathrm{Cone}\,(P,E_1)$, $\mathrm{Cone}\,(P,Q)$, $\mathrm{Cone}\,(P,S)$ where $Q = {}^t(0,c,1)$ and $S = {}^t(1,0,a)$. The divisor $E(P)$ is rational if and only if $\gcd(a,(bc-c+1)/e) = 1$ which is equivalent to $d = e$. We have $\det(P,E_1) = a > 1$, $\det(P,S) = a(c-1)/e > 1$ and $\det(P,Q) = (bc-c+1)/e$. As $\mathrm{Cone}\,(P,E_1)$ has e-copies of arms, $E(P)$ has at least three arms.

(V) Let $X_{\mathrm{V}} : x^a y + y^b z + z^c x = 0$. Then $P := {}^t(bc - c + 1, ca - a + 1, ab - b + 1)/d$ with $d = \gcd(bc - c + 1, ca - a + 1, ab - b + 1)$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 3 arms $\mathrm{Cone}\,(P,Q)$, $\mathrm{Cone}\,(P,S)$, $\mathrm{Cone}\,(P,T)$ where $Q = {}^t(0,c,1)$, $S = {}^t(1,0,a)$ and $T := {}^t(b,1,0)$. The divisor $E(P)$ is rational if and only if $d = 1$. In this case, we have $\det(P,Q) = bc - c + 1 > 1$, $\det(P,S) = ca - a + 1 > 1$ and $\det(P,T) = ab - b + 1 > 1$. Thus $E(P)$ has three arms.

(VII) Let $X_{\mathrm{VII}} : x^a z + y^b z + z^c + t x^{c_1} y^{c_2} = 0$. Then $P = {}^t(b(c-1), a(c-1), ab)/\delta$ with $\delta = \gcd(b(c-1), a(c-1), ab)$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 4 arms $\mathrm{Cone}\,(P,Q)$, $\mathrm{Cone}\,(P,S)$, $\mathrm{Cone}\,(P,E_1)$, $\mathrm{Cone}\,(P,E_2)$ where $Q = {}^t(0,1,c_2)$ and $S = {}^t(1,0,c_1)$. By the weighted homogenuity, we have the equality $b(c-1)c_1 + a(c-1)c_2 = abc$ which implies that $(c-1)|ab$. Hence $\delta = (c-1)\gcd(a,b,ab/(c-1))$. By loc. cit., $E(P)$ is rational if and only if either (i) $\gcd(a,b) = \gcd(a,c-1) = 1$, or (ii) $\gcd(a,b) = \gcd(b,c-1) = 1$. By symmetry, we may assume that the first case (i). Then $\delta = c-1$, $\det(P,Q) = b > 1$, $\det(P,S) = a > 1$, $\det(P,E_1) = a > 1$. Thus the resolution graph has at least three arms.

(VIII) Let $X_{\rm VIII}: x^ay+xy^b+xz^c+ty^{c_1}z^{c_2}=0$. Then $P={}^{\rm t}(c(b-1),c(a-1),b(a-1))/\delta$ with $\delta=\gcd(c(b-1),c(a-1),b(a-1))$. By the weighted homogenuity, we must have $c(a-1)c_1+b(a-1)c_2=c(ab-1)$ which implies that $(a-1)|c(ab-1)$ and $cc_1+bc_2=bc+c(b-1)/a-1$. Thus $\delta=(a-1)\gcd(b,c,c(b-1)/(a-1))$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 4 arms Cone (P,E_3), Cone (P,Q), Cone (P,S) and Cone (P,T) where $Q={}^{\rm t}(0,c,1)$, $S={}^{\rm t}(c_2,0,1)$ and $T={}^{\rm t}(c_1,1,0)$. The divisor $E(P)$ is rational if and only if $(b-1)=k(a-1)$ for some $k\in\mathbf{N}$ and $\gcd(b,c)=1$. Then $d=a-1$ and $\det(P,Q)=ck>1$, $\det(P,S)=c>1$, $\det(P,T)=b>1$ and $\det(P,E_3)=c$. Thus the $E(P)$ has at least 3 arms. □

4.4. **Normally smooth divisors on $T_{p,q,r}$-surfaces.** Let $T_{p,q,r}: x^p+y^q+z^r+xyz=0$ with $1/p+1/q+1/r<1$.

(1) Suppose that p,q,r are pairwisely coprime and $p<q<r$. The diagram $\Gamma^*(f)_2^+$ has three strictly positive vertices $P:={}^{\rm t}(rq-r-q,r,q)$, $Q:={}^{\rm t}(r,pr-p-r,p)$, and $R:={}^{\rm t}(q,p,pq-q-p)$. The cones Cone (P,E_1), Cone (Q,E_2) and Cone (R,E_3) are regular. Put $\delta:=pqr-pr-qr-pq$. Then $\det(P,Q)=\det(Q,R)=\det(P,R)=\delta$.

Proposition 23. *Under the above assumption, we have*

$$\rho(X_{p,q,r},O)=\rho_{QR}^{(1)}+\rho_{QR}^{(2)}+\rho_{QR}^{(3)}+\rho_{PR}^{(2)}+\rho_{PR}^{(3)}+\rho_{PQ}^{(3)}-2-\epsilon,$$

where $\varepsilon=1$ if $p=3$, and $\epsilon=0$ if $p\neq 3$.

Proof. This is a summary of the following three lemmas. □

Lemma 24. 1) $\mathcal{V}_{\rm ns}^{(1)}(Q,R)=\{P_k={}^{\rm t}(1,k,p-k-1)\mid p/q<k<(rp-r-p)/r\}$.
2) $\mathcal{V}_{\rm ns}^{(2)}(Q,R)=\{P'_k={}^{\rm t}(k,1,pk-k-1)\mid r/(pr-p-r)<k<q/p\}$.
3) $\mathcal{V}_{\rm ns}^{(3)}(Q,R)=\{P''_k={}^{\rm t}(k,pk-k-1,1)\mid q/(pq-p-q)<k<r/p\}$.
4) $\mathcal{V}_{\rm ns}^{(1)}(Q,R)\cap\mathcal{V}_{\rm ns}^{(2)}(Q,R)\cap\mathcal{V}_{\rm ns}^{(3)}(Q,R)\neq\emptyset$ *if and only if* $p=3$.
5) $\rho_{QR}=\rho_{QR}^{(1)}+\rho_{QR}^{(2)}+\rho_{QR}^{(3)}-1-\epsilon$, *where $\epsilon=1$ if $p=3$, and $\epsilon=0$ if $p\neq 3$.*

Proof. We mainly use Theorem 7. Let $P':=(\beta Q+\alpha R)/\delta={}^{\rm t}(p_1,p_2,p_3)$. The equation is

$$\begin{cases}\beta r+\alpha q=p_1\delta\\ \beta(pr-p-r)+\alpha p=p_2\delta\\ \beta p+\alpha(pq-p-q)=p_3\delta\end{cases}\quad\text{this implies}\quad\begin{cases}\alpha=(pr-p-r)p_1-rp_2\\ \beta=qp_2-pp_1\\ p_2+p_3=(p-1)p_1\end{cases}$$

Hence, we have the following conclusions.

1) $p_1=1$ if and only if there exists an integer $p_2>0$ such that $\alpha>0$ and $\beta>0$. This is equivalent to $p/q<p_2<(pr-p-r)/r$. And in this case $P'=(1,p_2,p-1-p_2)$.

2) $p_2=1$ if and only if there exists an integer $p_1>0$ such that $r/(pr-p-r)<p_1<q/p$. And in this case $P'=(p_1,1,(p-1)p_1-1)$.

3) $p_3=1$ if and only if there exists an integer $p_1>0$ such that $q/(pq-p-q)<p_1<r/p$. And in this case $P'={}^{\rm t}(p_1,pp_1-p_1-1,1)$.

4) is obvious now.

5) One can see this by comparing the three sets $\mathcal{V}_{\rm ns}^{(i)}(Q,R)$. In case $p=2$, we have $\mathcal{V}_{\rm ns}^{(1)}(Q,R)=\emptyset$ and $\mathcal{V}_{\rm ns}^{(2)}(Q,R)\cap\mathcal{V}_{\rm ns}^{(3)}(Q,R)=\{{}^{\rm t}(2,1,1)\}$. Hence, $\rho_{QR}=\rho_{QR}^{(2)}+\rho_{QR}^{(3)}-1$.

In case $p = 3$, we have $\mathcal{V}_{\rm ns}^{(i)}(Q,R)\cap\mathcal{V}_{\rm ns}^{(j)}(Q,R) = \mathcal{V}_{\rm ns}^{(1)}(Q,R)\cap\mathcal{V}_{\rm ns}^{(2)}(Q,R)\cap\mathcal{V}_{\rm ns}^{(3)}(Q,R) = \{{}^{\rm t}(1,1,1)\}$ for $i\neq j$. Hence, $\rho_{QP} = \rho_{QR}^{(1)}+\rho_{QR}^{(2)}+\rho_{QR}^{(3)}-2$.

In case $p > 3$, we have $\mathcal{V}_{\rm ns}^{(1)}(Q,R)\cap\mathcal{V}_{\rm ns}^{(2)}(Q,R) = \{{}^{\rm t}(1,1,p-2)\}$ and $\mathcal{V}_{\rm ns}^{(1)}(Q,R)\cap\mathcal{V}_{\rm ns}^{(3)}(Q,R) = \mathcal{V}_{\rm ns}^{(2)}(Q,R)\cap\mathcal{V}_{\rm ns}^{(3)}(Q,R) = \emptyset$. Hence, $\rho_{QP} = \rho_{QR}^{(1)}+\rho_{QR}^{(2)}+\rho_{QR}^{(3)}-1$. □

Similarly, one can prove the following two lemmas.

Lemma 25. 1) $\mathcal{V}_{\rm ns}^{(1)}(P,R) = \emptyset$.
2) $\mathcal{V}_{\rm ns}^{(2)}(P,R) = \{Q'_\ell = {}^{\rm t}(q-\ell-1,1,\ell)\mid q/r<\ell<(pq-p-q)/p\}$.
3) $\mathcal{V}_{\rm ns}^{(3)}(P,R) = \{Q''_\ell = {}^{\rm t}(q\ell-\ell-1,\ell,1)\mid p/(pq-p-q)<\ell<r/q\}$.
4) *Let* $Q' = {}^{\rm t}(q_1,q_2,q_3) = (\beta P+\alpha R)/\delta$. *Then* $(q-1)q_2 = q_1+q_3$.
5) $\rho_{PR} = \rho_{PR}^{(2)}+\rho_{PR}^{(3)}-1$. □

Lemma 26. 1) $\mathcal{V}_{\rm ns}^{(1)}(P,Q) = \mathcal{V}_{\rm ns}^{(2)}(P,Q) = \emptyset$.
2) $\mathcal{V}_{\rm ns}^{(3)}(P,Q) = \{R'_\ell = {}^{\rm t}(r-\ell-1,\ell,1)\mid r/q<\ell<(pr-p-r)/p\}$ *and* $\rho_{PQ} = \rho_{PQ}^{(3)}$. □

Example 27. (1) Let $p = 2, q = 3$ and $r\geq 7$. By the canonical subdivisions of the three cones, we see that $\rho_{QR} = \left[\frac{r-6}{2}\right]\geq 1$, $\rho_{PR} = \left[\frac{r-6}{3}\right]\geq 1$, and $\rho_{PQ} = \left[\frac{r-3}{6}\right]$.
(2) Let $p = 3, q = 4$ and $r > 4$. By the canonical subdivisions of the three cones, we see that $\rho_{QR} = \left[\frac{r}{3}\right]\geq 1$, $\rho_{PR} = \left[\frac{r}{4}\right]\geq 1$ and $\rho_{PQ} = [\frac{2r}{3}]-[\frac{r}{4}]-1$.

(2) Another case. Let $f(x,y,z) = x^n+y^n+z^n+xyz$ $(n\geq 4)$. The dual Newton diagram has three covectors $P_i, i = 1,2,3$ corresponding to the three compact faces. They are given by ${}^{\rm t}(n-2,1,1), {}^{\rm t}(1,n-2,1), {}^{\rm t}(1,1,n-2)$. And for $i\neq j$, $\det(P_i,P_j) = n-3$. Let $B_1,\dots,B_k$ be the vertices of the canonical subdivision of Cone(P_1,P_2) from P_1. Then $B_1 = (P_2+(n-4)P_1)/(n-3) = {}^{\rm t}(n-3,2,1)$. Thus $(n-3)/(n-4) = [2,\dots,2]$ with $(n-4)$-copies of 2. This implies $k = n-4$ and $B_j = {}^{\rm t}(n-2-j,1+j,1), j = 1,\dots,n-4$. In fact, by Lemma 5 the third coordinate of B_j is always 1 as both of P_1,P_2 have 1 as the third coordinate. Hence $\rho_{P_1P_2} = n-4$. The branch Cone(P_i,E_i) is regular. Thus $\rho(V,O) = \rho(\Sigma^*_{\rm can}) = 3n-9$ and *every exceptional divisor is normally smooth.*

5. Remarks

5.1. **Example of the inequality** $\rho(\Sigma^*_{\rm can}) > \rho(X,O)$**.** Let us consider A_{2c-1}-singularity, $X = \{x^2+y^2+z^{2c} = 0\}$. The resolution graph has two arms and the central divisor $E(P)$ is a rational curve with $E(P)^2 = -1$. Thus we have to blow-down the central divisor once (Example (6.7.1) in [9, III]). However in this example, the central exceptional divisor is not normally smooth, i.e., the extra blowing-up is line-admissible. So $\rho(\Sigma^*_{\rm can}) = \rho(X,O)$. The following gives an example of $\rho(\Sigma^*_{\rm can}) > \rho(X,O)$.

Example 28. Let X be defined by $h = xy+y^{bc}+z^c$ with $b,c\geq 2$. This is an A_{c-1}-singularity and a special case of $X_{\rm II}$ with $P := {}^{\rm t}(bc-1,1,b)$ and $Q := {}^{\rm t}(0,c,1)$.

Since $\det(P,E_1) = \det(P,E_3) = 1$ and $\det(P,Q) = bc-1$, we make the canonical subdivision of Cone(P,Q). The first covector T_1 from P is given by

$$T_1 = (Q+(bc-c-1)P)/(bc-1) = {}^{\rm t}(bc-c-1,1,b-1)$$

We have the continuous fraction expansion $(bc-1)/(bc-c-1) = [2, \dots, 2, 3, 2, \dots, 2]$ where the number of 2 in the first 2-series (respectively in the second 2-series) is $(b-2)$ (resp. $c-2$). Thus we have $c+b-3$ covectors $T_1, \dots, T_{b+c-3}$. The exceptional divisor $E(P)$ is rational with $E(P)^2 = -1$ and $E(T_j)$ with self intersection number $E(T_j)^2 = -2$ for $j \neq b-1$ and -3 for $j = b-1$ (see Theorem (6.3), Chapter III, [9]). In fact first $b-2$ covectors are given by

$$Q_j = {}^{\mathrm{t}}(cb - jc - 1, 1, b - j), \quad j = 1, \dots, b-1$$
$$Q_{b-1+j} = {}^{\mathrm{t}}(c - j - 1, j + 1, 1), \quad j = 1, \dots, c-2$$

and we see that they are normally minimal. To get a minimal reslution, we need to blow down $b-1$ divisors $E(P), E(T_1), \dots, E(T_{b-2})$ in this order. Then the self-intersection number of $E(T_{b-1})$ changes to -2 and we get A_{c-1} graph. In this example, we have $\rho(X, O) = c-1$ and $\rho(\Sigma^*_{\mathrm{can}}) = b + c - 2$.

5.2. **Parametrization of lines.** The normally smooth divisors on a surface X correspond to the lines on X. By using a toric resolution, one can give the exact parameterizations of the lines on X. This was done already for the Pham-Brieskorn surfaces in [4].

Proposition 29. *Suppose that we have a line L in a non-degenerate surface $X : f(x, y, z) = 0$ and assume that L is parametrized as*

$$x(t) = \alpha t^a + \alpha_1 t^{a+1} \dots, \quad y(t) = \beta t^b + \beta_1 t^{b+1} + \dots, \quad z(t) = \gamma t^c + \gamma_1 t^{c+1} + \dots$$

with $\alpha, \beta, \gamma \neq 0$ and $\min(a, b, c) = 1$. Let $P = {}^{\mathrm{t}}(a, b, c)$. Then the pull back of L intersects $E(P)$ transversally and $f_P(\alpha, \beta, \gamma) = 0$. Conversely any curve in $\mathcal{L}_{E(P)}$ has such a parametrization.

Example 30. (1) Let X be defined by $h = x^a y + y^b - z^b = 0$ with $a = a_1(b-1)$ and $a_1 > 1$. This is a special case of X_{II}. We use the notations in §4.2. Note that $P = {}^{\mathrm{t}}(1, a_1, a_1)$, $Q = {}^{\mathrm{t}}(0, b, 1)$, $\det(P, Q) = \det(P, E_3) = 1$ and $\det(P, E_1) = a_1$. By canonical subdivision of Cone (P, E_1) we have $R_i := {}^{\mathrm{t}}(1, i, i)$ with $i = 0, 1, \dots, i_1 = a_1$, where $R_0 := E_1$ and $R_{i_1} := P$. Hence $\rho_{PE_1} = a_1 - 1$. Since $r(P, E_1) + 1 = b$, each $E(R_i)$ has b components. By [9, III(6.3)], $E(P)^2 = -b < -1$. Hence π is minimal and $\rho(X, 0) = b(a_1 - 1) + 1$. The restriction of π on the toric chart associated with $\sigma_i :=$ Cone (R_i, R_{i-1}, E_2) is given by

$$\pi_{\sigma_i} : \quad x = uv, \quad y = u^i v^{i-1} w, \quad z = u^i v^{i-1}.$$

and the pull-back of h is given by

$$h \circ \pi_{\sigma_i} = u^{ib} v^{(i-1)b} \left(u^{(a_1-i)(b-1)} v^{(a_1-i+1)(b-1)} w + w^b - 1 \right)$$

The divisor $E(R_i)$ is defined by $u = 0$ and $w^b - 1 = 0$, hence $E(R_i)$ has b components. On this toric chart, the resolution $\tilde{X}$ of X is defined by

$$\tilde{h}_i(u, v, w) := u^{(a_1-i)(b-1)} v^{(a_1-i+1)(b-1)} w + w^b - 1 = 0$$

and in a neighborhood of $q \in E(R_i)$ we take u, v to be the local coordinates of $\tilde{X}$. Let $q = (0, s)$ in this coordinates. We consider the lines C_s defined by $t \mapsto (t, s)$. The image of C_s by π_{σ_i} is given by

$$\pi_{\sigma_i}(C_s): \quad x = st, \quad y = s^{i-1} w_k(t, s) t, \quad z = s^{i-1} t^i,$$

where $w_k(t, s)$ is the solution of $\tilde{h}_i(t, s, w) = 0$ with $w_k(0) = \exp(2\pi k i/b)$. As a special case, take $i = 1$. Then C_s is a normal line on $E(Q_1)$. When we moves $s \to 0$, this line approaches to $E(E_1)$ and $w_k(t) \equiv \exp(2k\pi i/b)$ and the image is the obvious line $t \to (x, y, z) = (0, w_k t, t)$.

(2) Let $X = T_{2,3,7} : x^2 + y^3 + z^7 + xyz = 0$. We have three covectors

$$P = {}^{\mathrm{t}}(11, 7, 3), \;\; Q = {}^{\mathrm{t}}(7, 5, 2), \;\; R = {}^{\mathrm{t}}(3, 2, 1)$$

and we do not need any other covector. Consider the toric chart $\sigma := (Q, R, E_3)$ with coordinates (u, v, w). Then the line $u = 1, v = t$ produces a line parametrized as $t \mapsto (t^3, t^2, -2t + 128t^2 + \dots)$.

5.3. **Obvious lines on surfaces.** We consider a surface $X = \{f(x, y, z) = 0\}$ where f has a non-degenerate Newton boundary. There are surfaces having obvious lines which can be read off from the polynomial defining the surface.

(1) Assume that $f(x, y, z)$ is not convenient and assume for example $\{y = z = 0\} \subset X$. Then as we have seen in Lemma 10, there is a unique non-compact face, different from the coordinate planes, which has the covector of the type $Q = {}^{\mathrm{t}}(0, c, 1)$ or ${}^{\mathrm{t}}(0, 1, c)$ and a unique covector P such that $\mathrm{Cone}\,(P, Q)$ is in $\Gamma^*(f)_2^+$ and P corresponds to a compact face. Let $Q_1, \dots, Q_k$ be the covectors defining the canonical regular subdivision from Q. Then Q_1 is a normally smooth divisor and $\mathcal{L}_{\mathrm{Q}_1}$ contains the canonical line $\{y = z = 0\}$.

(2) Assume that $h(x, y) := f(x, y, 0)$ (the section of f with $z = 0$) is a non-monomial homogeneous polynomial of degree d. Then we can factor $h(x, y) = c x^a y^b \prod_{i=1}^k (y - \alpha_i x)$. Thus X has the lines $z = 0, y = \alpha_i x$ for $i = 1, \dots, k$. Combinatorially this says the following. There exists a compact face Δ such that $\Delta \supset \Delta(h)$. The corresponding covector takes the form $P = {}^{\mathrm{t}}(p, p, r)$ with $\gcd(p, r) = 1$. Then the first covector Q_1 from E_3 in the canonical regular subdivision of $\mathrm{Cone}\,(P, E_3)$ takes the form $Q_1 = {}^{\mathrm{t}}(1, 1, s)$ with $s = 1 + [r/p]$. So we can see that $Q_1 \in \mathcal{V}_{\mathrm{ns}}(P, E_3)$. A typical example is $T_{n,n,n} : x^n + y^n + z^n - xyz = 0$. Another example is (1) of Example 30.

(3) Assume that the monomial x^A in f such that $(A, 0, 0) \in \Gamma(f)$. We say that x^A is negligibly truncatable if $f_t(x, y, z) = (f(x, y, z) - f(x, 0, 0)) + t f(x, 0, 0)$ defines a μ-constant family for $0 \le t \le 1$ (cf. [11]). Assume for example, the monomials $x^a y$ and $x^b z^c$ are on the non-compact face of $\Gamma(f_0)$. Let $Q' := {}^{\mathrm{t}}(c/d, c(A - a)/d, (A - b)/d)$ with $d = \gcd(c, A - b)$. The covector Q' corresponds to the negligible compact face of f_1 containing $(a, 1, 0), (b, 0, c)$ and $(A, 0, 0)$. Then there is a normally smooth divisor on $\mathrm{Cone}\,(Q, E_3)$. In fact, $\det(Q', E_3) = c/d$. If $c = d$, Q gives normally smooth divisor. If $c > d$, the first covector Q_1' of the canonical regular subdivision of $\mathrm{Cone}\,(Q', E_3)$ is normally smooth. An example is given by $f(x, y, z) = x^2 y + y^2 + z^5 + x^5$. Then x^5 is negligibly truncatable.

(4) Assume that $\Gamma(f)$ has a compact face whose covector P has 1 in its coefficients. Then $E(P)$ is a normally smooth divisor. This is the case, for example, if $P = {}^t(1,1,1)$ and $f_P(x,y,z)$ has a two-dimensional support. We can see easily that $E(P)$ is isomorphic to the projective curve $f_P(x,y,z) = 0$ in $\mathbb{P}^2$. The tangent cone of X at O is given by the cone of $f_P = 0$.

5.4. **Normally smooth divisors on complete intersections.** In this paper we mainly considered normally smooth divisors on two dimensional hypersurface singularities. However every assertion can be generalized to non-degenerate complete intersections. We give an example. Consider the surface given by $X = \{f_1(x,y,z,w) = f_2(x,y,z,w) = 0\}$ where f_1 and f_2 has the same Newton boundary. Assume that f_1, f_2 are Pham-Brieskorn polynomials of the same type, with generic coefficients:

$$f_i = a_i x^{p_1} + b_i y^{p_2} + c_i z^{p_3} + d_i w^{p_4}, i = 1, 2$$

We assume that $p_1, \dots, p_4 \geq 2$ and mutually coprime. Then the dual Newton diagram $\Gamma^*(f_1, f_2)$ is the same with $\Gamma^*(f_i)$ and $\Gamma^*(f)_2^+$ is star-shaped with the center $P = {}^t(p_2p_3p_4, p_1p_3p_4, p_1p_2p_4, , p_1p_2p_3)$ and four arms $\mathrm{Cone}\,(P, E_i), i = 1, \dots, 4$. We consider the $\mathrm{Cone}\,(P, E_1)$. First $\det(P, E_1) = p_1$. By Lemma 11, $\mathcal{V}_{ns}^{(i)}(P, E_1) = \emptyset$ for $2 \leq i \leq 4$. As for $\mathcal{V}_{ns}^{(1)}(P, E_1) \neq \emptyset$ if and only if $p_2p_3p_4 < p_1$ and putting $r = [p_1/p_2p_3p_4]$, $\mathcal{V}_{ns}^{(1)}(P, E_1) = \{Q_j = (jP + (p_1 - jp_2p_3p_4)E_1)/p_1; j = 1, \dots, r\}$.

References

[1] G. Gonzalez-Springberg, M. Lejeune-Jalabert, Courbes lisses sur les singularités de surface, *C. R. Acad. Sci. Paris, t. Sétie, I*, **318** (1994), 653-656.

[2] G. Gonzalez-Springberg, M. Lejeune-Jalabert, Families of smooth curves on surface singularities and wedges, *Ann. Polonici Math.,* LXVII.2 (1997), 179-190.

[3] G. Jiang, *Functions with non-isolated singularities on singular spaces,* Thesis, Universiteit Utrecht, 1998.

[4] G. Jiang, M. Oka, D. T. Pho, D. Siersma, Lines on Brieskorn-Pham surfaces, *Kodai Math J.* to appear.

[5] G. Jiang, D. Siersma, Local embeddings of lines in singular hypersurfaces, *Ann. Inst. Fourier (Grenoble)*, **49** (1999) no. 4, 1129–1147.

[6] A. G. Khovanskii, Newton polyhedra and the genus of complete intersetions, *Funkts. Anal. Prilozhen.* **12** (1977) no. 1, 51-61.

[7] A.G. Kouchnirenko, Polyèdres de Newton at Nombres de Milnor, *Invent. Math.* **32** (1976) no. 1, 1-31.

[8] H. B. Laufer, Normal two-dimensional singularities, *Ann. Math. Studies,* **71**, Princeton Univ. Press, Princeton 1971.

[9] M. Oka, Non-degenerate complete intersection singularity, Hermann, Actualité en Mathématiques, 1997.

[10] M. Oka, On the resolution of the hypersurface singularities, in: eds. T. Suwa, P, Wagreich, Complex analytic singularities, *Advanced Studies in Pure Math.* **8** (1987), Kinokuniya Company, Tokyo, p. 405-436.

[11] M. Oka, On the simultaneous resolution of a negligible truncation of the Newton boundary, *Singularities, Contempolary Mathematics,***90**, Amer. Math. Soc. (1989), 199-210.

[12] P. Orlik, P. Wagreich, Isolated singularities of algebraic surfaces with $\mathbb{C}^*$-action, *Annals of Math.* **93** (1971), 205-228.

Jiang Guangfeng
Department of Mathematics and Physics
Faculty of Science
Beijing University of Chemical Technology
Beijing, 100029
P. R. China

Mutsuo Oka
Department of Mathematics
Tokyo Metropolitan University
1-1 Mimami-Ohsawa, Hachioji-shi
Tokyo 192-0397